EFFECT OF SURFACE CURVATURE IN RAREFIED-GAS TRANSFER PROCESSES

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The characteristics of momentum and kinetic energy transfer in rarefied gases are analyzed on the basis of the elementary kinetic theory of gases. The proposed formulas take into account the discreteness of the gas structure which, as is shown, is manifested in the presence of curvature of the heat-transfer surfaces. The validity of the formulas is confirmed by existing experimental data.

It is one of the basic principles of the kinetic theory of gases that the viscosity and thermal conductivity of gases do not depend on pressure. This conclusion is suggested by the very structure of the expressions for the viscosity [1]

$$\eta = 0.499\rho vL, \tag{1}$$

where $\rho = mn$; $L = kT/\sqrt{2}\pi\sigma^2 p = 1/\sqrt{2}\pi\sigma^2 n$, and the thermal conductivity

$$\lambda = \varepsilon\eta c_v, \tag{2}$$

where $\varepsilon = (9\gamma - 5)/4$; $\gamma = c_p/c_V$. In these expressions, the product of the density ρ and the mean free path L is approximately constant ($\rho L \sim$ const), and therefore $\eta \sim \lambda \sim \rho L$ over a broad range of pressure variation.

The numerical values of the coefficients of viscosity η and thermal conductivity λ can be determined by measuring the viscous stresses τ and the heat flux q between two plane surfaces. The corresponding expressions take the form [1]

$$\tau = \frac{\eta u}{d} \tag{3}$$

$$q = \frac{\lambda (T_1 - T_2)}{d}. \tag{4}$$

However, at pressures at which $L > d$ the effective values of the viscosity and thermal conductivity are reduced in view of the limitations on the increase in L with simultaneous decrease in the density ρ. In fact, as has been experimentally established [1, 2], the reduction in the rate of transfer of momentum and kinetic energy with decrease in pressure is observed at a much higher pressure than that corresponding to the condition $L > d$. This is equivalent to a decrease in the transfer coefficients—viscosity and thermal conductivity. Qualitatively, the effect is attributed [1–8] to the slip of the gas molecules at the surface and the temperature jump at the gas-wall interface. Quantitatively, the effects are taken into account by introducing into Eqs. (3) and (4) a correction for the discrete nature of the medium, whereupon these equations respectively take the form [1, 4, 5]

$$\tau_s = \frac{\eta u}{d + 2\xi}, \tag{5}$$

where $\xi = 2 \cdot 0.499L[(2 - f)/f]$;

$$q_j = \frac{\lambda (T_1 - T_2)}{d + 2g}, \tag{6}$$

where $g = ((2 - \alpha)/\alpha)(2/(\gamma + 1))(\lambda/\eta c_V)L = \beta L$.

In view of the fact that the momentum transfer (f) and thermal accommodation (α) coefficients vary between 0 and 1, the slip (ξ) and temperature jump (g) coefficients are, in the general case, indeterminate. As a rule, their numerical values are found experimentally.

In one of the few studies of the temperature jump and its consequences for the one-dimensional problem Lazarev [3] recorded the temperature field between two plane plates separated by a distance $d = 9 \cdot 10^{-3}$ m. At pressures $p = 8.7 \cdot 10^{-2}$ mm Hg and below ($Kn = L/d \geq 0.07$) he experimentally detected, if not a temperature jump in the normal sense, a marked deformation of the temperature field equivalent to a jump in a layer of gas with thickness on the order of the mean free path adjacent to the plates.

The experimental determination of the temperature jump in heat transfer between two parallel plates is associated with known experimental difficulties, because in rarefied gases the fluxes are small and the conditions of one-dimensionality are difficult to satisfy. It is for precisely this reason that in studying heat transfer in rarefied gases the methods of concentric spheres and coaxial cylinders are used [1–3, 6–8]. For the case of two coaxial cylinders, the Fourier heat conduction equation takes the form

$$q^{cy} = \frac{\lambda (T_1 - T_2)}{r_0 \ln (R/r_0)}, \tag{7}$$

and with allowance for the temperature jump

$$q_j^{cy} = \frac{\lambda (T_1 - T_2)}{r_0 \ln (R/r_0) + \beta L[(r_0/R) + 1]}. \tag{8}$$

Apart from the fact that expression (8) still lacks strict physical verification, in practice it is inapplicable to heat transfer in an infinite space (as $R \to \infty$) and can be used only for a qualitative analysis at large values of R/r_0. Moreover, both in the earlier work of Dulong and Petit, Kundt and Warburg [1, 3], and Smoluchowski [1, 2] as well as in the later studies of Kyte, Madden, and Piret [6], effects equivalent to a temperature jump were observed at higher pressures, but for heat transfer between surfaces other than plane.

On the other hand, the use of the Knudsen number ($Kn = L/d$) for estimating the applicability of the continuum theory is not sufficiently rigorous or well founded, in view of the indeterminacy of the characteristic dimension d. In fact, the quantity b may be both the distance between the plane heat-transfer surfaces and the diameter d_0 of the inner cylinder or sphere, if heat transfer takes place between two coaxial cylinders or concentric spheres. At the same time, in practice it is more usual to encounter systems of arbitrary shape and the choice of a characteristic dimension becomes difficult.

This situation cannot be regarded as satisfactory; hence, the need to refine the model and mechanism of energy transfer in rarefied gases.

Without disputing the presence of slip and a temperature jump at the gas-solid interface, but assuming that they are negligibly small, we can show that the reduction in the rate of heat transfer between two coaxial cylinders or spheres with decrease in pressure is attributable to the curvature of the heat-transfer surfaces.

First, we consider the nature of the qualitative difference in heat transfer through a rarefied gas for parallel plates and coaxial cylinders. Let all the heat-transfer conditions be perfectly identical, except one—the absence of curvature of the surfaces in the first case and its presence in the second. Thus, to preserve the identity, the gases, their temperatures and their thermophysical properties must be the same in both cases. Moreover, the distances between the heat-transfer surfaces should be equal

$$d = R - r_0, \tag{9}$$

as should the absolute temperatures of like surfaces.

To examine the effect of the curvature of the surfaces on the heat transfer between them, we transform Eqs. (4) and (7), using (1) and (2), so as to isolate the complexes that do not depend on curvature. Then Eq. (4) is written in the form

$$q = \frac{\lambda}{L} \Delta T_L, \tag{10}$$

where

$$\Delta T_L = \frac{T_1 - T_2}{d} L \tag{11}$$

is the temperature drop over one mean free path L (Fig. 1a). At small temperature gradients, neglecting the temperature dependence of the mean free path, we may assume that $\Delta T_L \approx$ const.

We obtain analogous relations for coaxial cylinders from Eq. (7):

$$q^{cy} = \frac{\lambda}{L} \Delta T_L^{cy}, \tag{12}$$

where

$$\Delta T_L^{cy} = \frac{T_1 - T_2}{r_0 \ln (R/r_0)} L. \tag{13}$$

In (10) and (12), in view of the conditions of the problem, the complexes λ/L are the same.

Comparing (11) and (13), we find that as a result of the inequality $r_0 \ln(R/r_0) < R - r_0$ or, using (9), $r_0 \ln(R/r_0) <$ $< d$ the inequality

$$\Delta T_L^{cy} > \Delta T \tag{14}$$

is always satisfied, from which it follows that if all of the identity conditions are satisfied except one—the curvature of the surfaces, for a continuum the heat flux density is always greater for coaxial cylinders than for flat plates.

Further analysis of Eqs. (11) and (13) shows that even in the presence of curvature of the surfaces it is possible to find a combination of physical parameters of the gas for which the identity conditions are satisfied exactly. In fact, let the relations between L and d in (11) be such

$$L = d, \tag{15}$$

that for the one-dimensional problem $\Delta T_L = T_1 - T_2$. Then to ensure identicalness it is necessary to require that in Eq. (13) the equality $\Delta T_L^{cy} = T_1 - T_2$ or

$$\Delta T_L = \Delta T_L^{cy} \tag{16}$$

be satisfied, which is formally possible only when

$$L = r_0 \ln (R/r_0). \tag{17}$$

The latter is physically impracticable owing to the previously adopted condition (9). The only parameter that can be varied in order to preserve all the identity conditions is the pressure of the gas p.

In fact, considering relation (1), for a layer of gas of thickness $R - r_0$ we can select a pressure

$$p_r > p, \tag{18}$$

such that all the identity conditions (9), (15), (16) and (17) are satisfied. From the latter inequality it follows that the curvature of the heat-transfer surfaces affects the energy transfer process in the same way as the gas pressure. Consequently, for each change in the curvature of the surfaces between which transfer processes are observed there is an equivalent change of pressure. It is in this that the qualitative originality of energy transfer processes in the presence of curvature of the surfaces essentially consists. We now show to what quantitative changes this leads in the process of heat transfer by conduction through a rarefied gas.

In our qualitative analysis we did not impose any limitations either on the thermophysical parameters of the gas T, η, λ, L, or on the geometry of the heat-transfer systems d, r_0, R. Therefore, our conclusions are valid for any values of T, η, λ, L, d, r_0, R. It is only important to satisfy the identity conditions.

With this in mind, we find an expression for the temperature drop ΔT_1 in a layer of gas immediately adjacent to the surface of the inner cylinder and one mean free path thick (Fig. 1b). Then, assuming that at L the interaction process corresponds to total energy transfer between the colliding molecules, by analogy with (13) we write

$$\Delta T_1 = \frac{T_1 - t_2}{r_0 \ln [(r_0 + L)/r_0]} L, \tag{19}$$

where t_2 is the temperature of the gas at the surface of an imaginary cylinder of radius $r_0 + L$.

Similar values of the temperature drops $\Delta t_2, \ldots, \Delta t_n, \ldots, \Delta T_2$ are found for the other layers of thickness L:

$$\Delta t_2 = \frac{t_2 - t_3}{(r_0 + L) \ln [(r_0 + 2L)/(r_0 + L)]} L, \tag{20}$$

. . . .

$$\Delta t_n = \frac{t_n - t_{n+1}}{[r_0 + (n-1)L] \ln \{(r_0 + nL)/[r_0 + (n-1)L]\}} L, \tag{21}$$

. . . .

$$\Delta T_2 = \frac{t_2' - T_2}{(R - L) \ln [R/(R - L)]} L, \tag{22}$$

where ΔT_2 is the temperature drop in a layer of gas of thickness L immediately adjacent to the inner surface of the outer cylinder.

Expressions (19)–(22) were obtained on the same initial assumptions as expression (13). Therefore, they can be used for calculating the heat flux equally with (13). Below it will be shown that the advantage of expressions (19)–(22) consists in the possibility of using them to take into account the effect of the curvature of the heat transfer surfaces.

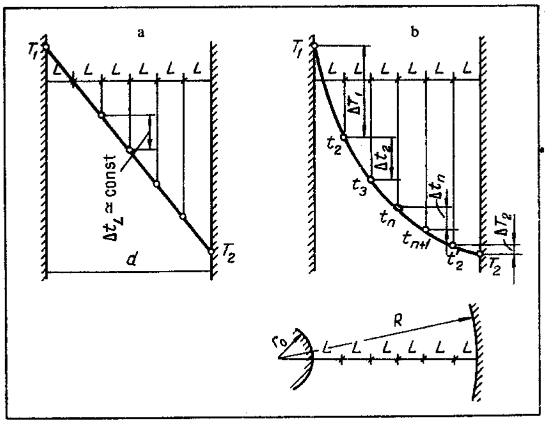

Fig. 1. Model of molecular transfer processes: a) between plane surface; b) between coaxial cylinders or spheres.

Following the ideas developed in our qualitative analysis, to satisfy the identity conditions we must find within each layer of thickness L a series of equivalent pressures, whose values satisfy the inequality

$$p_{r_1} > p_{r2} > \ldots > p_{rn} > \ldots > p_R. \tag{23}$$

In reality there is no pressure gradient satisfying inequality (23). Accordingly, to preserve identity conditions (9), (15), (16) and (17) we can proceed by selecting an equivalent thickness of the layer of gas L_r, for which condition (16) is satisfied. Establishing in Eqs. (19)–(22), respectively, the following equalities: $\Delta T_1 = T_1 - t_2$; $\Delta t_2 = t_2 - t_3, \ldots, \Delta t_n = t_n - t_{n+1}, \ldots, \Delta T_2 = t_2' - T_2$, we have

$$L_{r1} = r_0 \ln [(r_0 + L)/r_0], \tag{24}$$

$$L_{r2} = (r_0 + L) \ln [(r_0 + 2L)/(r_0 + L)], \tag{25}$$

. . . .

$$L_{rn} = [r_0 + (n-1)L] \ln \{(r_0 + nL)/[r_0 + (n-1)L)\}, \tag{26}$$

. . . .

$$L_R = (R - L) \ln [R/(R - L)]. \tag{27}$$

The values of L_r obtained can be used as the equivalent of L in (13), if we aim at finding local values ΔT_L^r (for each layer of thickness L) and not ΔT_L^{cy}, averaged over the entire thickness of the $R - r_0$ layer. The general pattern of the temperature field is not disturbed by substituting L_r for L, since $L_r = \varphi[L, \ln(r + L)/r]$, where $r_0 \le r \le (R - 2L)$, and the local values of L_r satisfy the inequality $L_{r_1} < L_{r_2} \ldots < L_{r_n} < \ldots < L_R$.

In practical calculations the parameters L_{r1} and L_R may be of importance. The former may be regarded as the thickness of the boundary layer on the surface of the inner cylinder and used to calculate the heat losses per unit surface of the inner cylinder. The parameter L_R may be regarded as the thickness of the boundary layer on the surface of the outer cylinder; it can be used to calculate the inflow of heat per unit surface of the outer cylinder.

Replacing the parameter L and L_{r1}, in accordance with (24), in Eq. (13) with subsequent substitution of the result into Eq. (12) gives the value of the heat losses q_c^{cy} with allowance for the curvature of the surface of the inner cylinder

$$q_c^{cy} = \frac{\lambda(T_1 - T_2)}{\ln(R/r_0)} \frac{\ln[1 + (L/r_0)]}{L}. \quad (28)$$

We obtain a similar formula for the specific heat flux absorbed by the outer cylinder if in (13) we replace L with L_r from (27):

$$q_{cR}^{cy} = \frac{\lambda(T_1 - T_2)(R - L)\ln[R/(R - L)]}{\ln(R/r_0)\,L}. \quad (29)$$

In both cases, passage to the limit as $L \to 0$ gives the usual equation of heat conduction (7) for a continuum, since

$$\lim_{L \to 0} \frac{r_0 \ln[1 + (L/r_0)]}{L} = 1 \quad \text{and} \quad \lim_{L \to 0} \frac{(R - L)\ln[R/(R - L)]}{L} = 1.$$

Performing a similar analysis of the continuum equation for concentric spheres [6]

$$q^{sp} = \frac{\lambda(T_1 - T_2)}{r_0 - (r_0^2/R)}, \quad (30)$$

we obtain

$$q_c^{sp} = \frac{\lambda(T_1 - T_2)}{L[r_0 - (r_0^2/R)]} L_r. \quad (31)$$

After constructing equations such as (19)–(22), but for concentric spheres a distance L apart, extracting the parameter L_r and substituting values of L_r into Eq. (31), we have

$$q_c^{sp} = \frac{\lambda(T_1 - T_2)}{r_0 - (r_0^2/R)} \frac{1}{[1 + (L/r_0)]}, \quad (32)$$

$$q_{cR}^{sp} = \frac{\lambda(T_1 - T_2)}{r_0 - (r_0^2/R)} \frac{1}{1 + (L/R)}, \quad (33)$$

where the multipliers $\psi_1 = 1/[1 + (L/r_0)]$ and $\psi_2 = 1/[1 + (L/R)]$ take into account the effect of the curvature of the spheres. As $L \to 0$ $\psi_1 = \psi_2 \to 1$.

To confirm the validity of formula (28) and estimate its accuracy, we compared the calculations with the most reliable experimental data [6] on the heat transfer of a platinum wire 0.078 mm in diameter in air at pressures from 0.1 to 760 mm Hg. As the outer radius we took the distance between the axis of the wire and the resistance thermometer (R = 152 mm). As the characteristic temperature we took the mean integral temperature in the gas volume between the inner and outer cylinders:

$$T_m = T_2 + \xi_1 \Delta T,$$

where

$$\xi_1 = \frac{1}{2}\left\{2 - \left[\frac{2}{1-(r_0/R)^2} - \frac{1}{\ln(R/r_0)}\right]\right\}.$$

The thermal conductivity of air λ was calculated with allowance for its temperature dependence [1], and the mean free path L at the surface temperature of the wire T_1.

The heat losses from the platinum wire to the gas, calculated from (28), are represented by the solid line in Fig. 2, which gives the heat transfer coefficient $\alpha_c^{cy} = q_c^{cy}/(T_1 - T_2)$ as a function of pressure. The points represent the experimental values of the heat-transfer coefficient [6].

The calculated curve corresponds, with an error not exceeding 10%, to the experimental values in a pressure region on the order of 10–500 N/m² ($0.2 < Kn_0 < 10$), which indicates the validity of Eq. (28) in the pressure region in which convection is practically absent. To take the effect of convection into account at pressures of 500 N/m² and above, we made a calculation of the boundary layer as a function of pressure in accordance with Langmuir's formula [1]

$$r \ln(r/r_0) = B, \tag{34}$$

where $B = CL(T_m/M)^{1/2}$; for air $B = 4.3 \cdot 10^{-3}$ m.

In this case taking convection into account reduces to finding the outer radius r of the cylindrical boundary layer and substituting the corresponding value in (28) instead of R. The calculated curve, corrected for the effect of convection, is represented by the broken line in Fig. 2. The discrepancy between calculation and experiment is within 20–25%. Obviously, it is a consequence of disregarding the vertical orientation of the wire and the temperature jump at its surface, on the one hand, and the approximate nature of the Langmuir formula (34), on the other. Comparison of theory and experiment [6] for a sphere 7.9 mm in diameter ($10^{-3} < Kn_0 < 10^{-1}$) gives qualitatively good agreement.

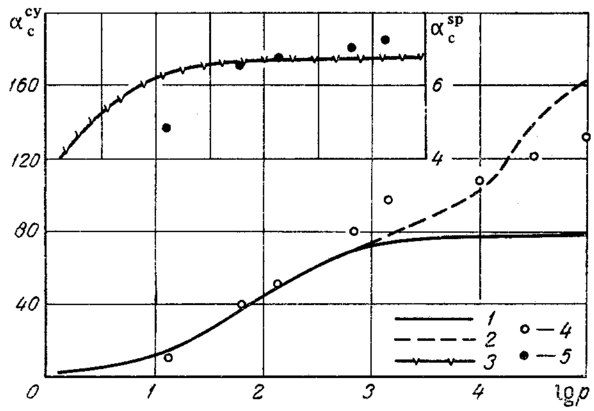

Fig. 2. Heat-transfer coefficients α_c^{cy}, α_c^{sp} (W/m² · deg) as functions of rarefied air pressure in the presence of curvature of the heat-transfer surfaces (p, N/m²): 1) theory for heat conduction between coaxial cylinders; 2) the same with allowance for convection; 3) theory for spheres; 4) experimental for cylinders; 5) the same for spheres.

To refine relations of type (28), (32) and to take into account certain other subtleties of transfer processes in rarefied gases, it is necessary to organize special experiments in the light of the theory expounded above. Nonetheless, the very fact of experimental confirmation of a theory based on an idea completely unrelated to the concepts of gas slip and temperature jumps provides a basis for reexamining certain results of transport theory in rarefied gases following from modern views concerning the importance of the discrete structure of the medium. In particular, this applies to the experimental data on the accommodation coefficient obtained from the temperature jump at the surface of cylindrical wires.

NOTATION

k is Boltzmann's constant; σ is the diameter of the gas molecule; q is the specific heat flux; T and t are temperatures; p is the pressure; η and λ are the coefficient of viscosity and the thermal conductivity of the gas, respectively; c_p and c_v are the specific heat at constant pressures and constant volume; ρ and u are the density and mass-averaged velocity of the gas; m, $\bar{v}$, L are the mass, the mean velocity, and the mean free path of the gas molecules; f and α are the diffuse reflection and thermal accommodation coefficients; r_0, R, and r are the inner and outer radii and the variable radius of the coaxial cylinders or spheres; $d_0 = 2r_0$; d is the distance between two parallel plates; $Kn_0 = L/d_0$ or $Kn = L/d$ is the Knudsen number. Indices: cy—cylinder; sp—sphere; 1, r1—inner; 2, R—outer; m—mean; j—jump; s—slip; c—curvature.

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